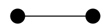
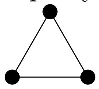
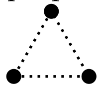
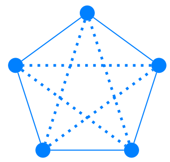
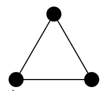
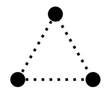
Chapter 9 Ramsey Theory (complete chaos is not possible)

Suppose people at a party. Two know each other  or don't know each other 

1: Find a diagram of a party of 5 people such that no 3 people all know each other or do not know each other. That is, we don't see  or 

Solution:



2: Is it possible to find a diagram on 6 people without  and  ?
(Hint: Pick one person and investigate who she knows. . .)

Solution: Pick person x. Without loss of generality assume he knows at least 3 other people. The these 3 either 1 pair know each other or non know each other - triangle is in both cases.

Notation: $K_6 \to K_3K_3$ reads as "K_6 arrows K_3K_3" and means in every coloring of edges of K_6 by two colors, there exists either K_3 in the first color or K_3 in the second color.

Notice that edges and non-edges can be treated as 2 colors.

Theorem (Ramsey) $\forall m, n, \exists p$ such that $K_p \to K_mK_n$.

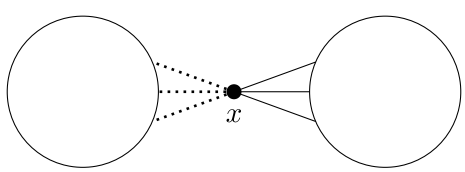
In other words, every 2-coloring of a huge graph K_p contains a monochromatic K_m or K_n.

Denote smallest p by $r(m, n)$.

3: Determine $r(2, n)$ and $r(1, n)$.

Solution: $r(1, n) = 1$ and $r(2, n) = n$. If one edge, is red, we have red K_2. If all edges are blue, we have blue K_n. It cannot be $< n$, otherwise all edges blue would not give red K_2 or blue K_n.

4: Show that $r(m, n) \leq r(m-1, n) + r(m, n-1)$. (Hint: Consider $p = r(m-1, n) + r(m, n-1)$ points. Pick any point x and study set of blue or red neighbors.)



Solution: By pigeonhole principle, either first part has size $r(m-1, n)$ or the second has size $r(m, n-1)$ (by pigeonhole principle). In either case, the result and x guarantee monochromatic K_m or K_n.

Corollary The Ramsey Number $r(m,n)$ exists and $r(m,n) \le \binom{m+n-2}{m-1}$.

5: Prove the corollary by induction.

Solution: The proof is by induction on $m+n$. If $m+n = 2,3$, then this is obviously true.

$$r(m,n) \le r(m-1,n) + r(m,n-1) = \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1}$$

Hence by Pascal's identity

$$r(m,n) \le \binom{m+n-2}{m-1}.$$

Calculating $r(m,n)$ exactly is actually a very hard problem even for small m,n.
See `https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1/pdf`

	3	4	5	6	7
3	6	9	14	18	23
4		18	25	36–41	49–61
5			43–48	58–87	80–143
6				102–165	115–298

Ramsey's theorem can be extended to more than 2 colors. For c colors, we have $K_p \to K_{n_1} K_{n_2} \cdots K_{n_c}$.

6: Show Ramsey's theorem for 3 colors. That is, prove that $r(m,n,o)$ is finite (minimum p such that $K_p \to K_m K_n K_o$).

Solution: The trick is to show $r(m,n,o) \le r(m, r(n,o)) := n_r$. First imagine that n and o being just one color. Let φ be a 3-edge-coloring if K_{n_r}. If φ contains K_m in the first color, we are done. Otherwise φ contains $K_{r(n,o)}$ in the second and third color. By Ramsey's theorem for 2 colors, this $K_{r(n,o)}$ contains a monochromatic K_n or K_o. This argument easily generalizes to more colors.

Let $G_1, G_2, \ldots, G_r$ be graphs. The Ramsey number, $r(G_1, G_2, \ldots, G_r)$ is the smallest n such that every r-edge coloring of K_n contains a monochromatic subgraph isomorphic to G_i in color i for some $i \in 1 \ldots r$.

7: Show that $r(G_1, G_2, \ldots, G_r)$ is finite.

Solution: Let $|V(G_i)| = k_i$. Then $r(G_1, G_2, \ldots, G_r) \le r(K_{k_1}, K_{k_2}, \ldots, K_{k_r})$

For particular choices of G_i, the Ramsey number may be smaller than exponential.

Proposition Let s,t be positive integers, and let T be a tree on t vertices. Then $r(T, K_s) = (s-1)(t-1)+1$.

8: Find a 2-coloring of $K_{(s-1)(t-1)}$ without monochromatic copy of T in the first color and a monochromatic copy of K_s in the second color. Or find a graph on $(s-1)(t-1)$ vertices that does not contain T as a subgraph and $\alpha(G) \le s-1$.

Solution: Take $s-1$ disjoint copies of K_{t-1}. Notice that this does not contain T and $\alpha(G) = s-1$.

9: Show that any graph G on $(s-1)(t-1)+1$ vertices with $\alpha(G) \leq s-1$ contains T as a subgraph.
Hints: What is $\chi(G)$? Can you find a subgraph of G with a large minimum degree? Can you find T in it?

Solution: Because $\alpha(G) \leq s-1$, $\chi(G) \geq ((s-1)(t-1)+1)/(s-1) \geq t$. In every graph G with $\chi(G) \geq t$ exists a subgraph with minimum degree $t-1$, why? Take such subgraph H of G. Now every graph with minimum degree $t-1$ contains any tree on t vertices. It can be found simply by a greedy embedding algorithm. Hence H contains T as a subgraph.

Ramsey's theorem can be extended to coloring more than pairs of vertices. For c colors, we have

$$K_p^t \to K_{n_1}^t K_{n_2}^t \cdots K_{n_c}^t,$$

which means that if we color all t subsets of vertices by c colors, there exists i such that there are n_i vertices where all t-subsets have color i.

One could do even an infinite version of the Ramsey's theorem.

Theorem (Infinite Ramsey)
Let k, c be positive integers, and X an infinite set. If each k-subset of X is colored with one of c colors, then X has an infinite monochromatic subset.

Proof The proof goes by induction on k.

10: Show the theorem is true for $k = 1$

Solution: We are coloring infinitely elements by c colors. Hence there must be one color-class that has infinitely many points in it.

Now assume that each k-subsets of X are colored by one of c colors. We will create an infinite sequence $X_0, X_1, \ldots$ of infinite subsets of X and choose elements $x_i \in X_i$ witht the following properties

1. $X_{i+1} \subseteq X_i \setminus \{x_i\}$
2. all k-subsets $\{x_i\} \cup Z$ with $Z \subseteq X_{i+1}$ have the same color, which we associate to x_i.

11: Show that if we have the sequence $X_0, X_1, \ldots$ and $x_0, x_1, \ldots$, it is easy to finish the proof.

Solution: There are finitely many colors. Hence exists infinite $Y \subseteq \{x_0, x_1, \ldots\}$ where all elements in Y have the same associated color. That means every k-subset in Y has the same color.

12: Show how to iteratively build $X_0, \ldots$ using induction on k.
Hint: Start $X_0 := X$ and pick x_0 arbitrarily. How to get X_1?

Solution: Color each $k-1$ subset Z of $X_i \setminus \{x_i\}$ by the color of $Z \cup \{x_i\}$. By induction, there exists an infinite monochromatic subset of $X_i \setminus \{x_i\}$ for this coloring of $k-1$ sets. We call it X_{i+1} and pick x_{i+1} in X_{i+1} arbitrarily.

Probabilistic lower bound by Erdős $r(k,k) \geq \lfloor 2^{k/2} \rfloor$ for all $k \geq 3$.

Consider a random coloring of edges of K_n by red and blue. That is for each edge of K_n we choose independently uniformly at random if the edge is red or blue.

13: What is the number of edges of K_n?

Solution: $\binom{n}{2}$

14: What is the probability that a fixed set of k vertices induces a red clique? (all edges are red)

Solution: $\frac{1}{2^{\binom{k}{2}}} = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}}$

15: What is the probability that a fixed set of k induces a monochromatic clique? (all edges red or blue)

Solution: $2^{1-\binom{k}{2}}$

16: What is the possible number of k-subsets? (candidates for monochromatic cliques)

Solution: $\binom{n}{k}$

17: What is the expected number of monochromatic subsets of size k?
Recall expected value of X is $EX = \sum_X p(X)X$.

Solution: $\binom{n}{k}2^{1-\binom{k}{2}}$

18: Try to use $n = \lfloor 2^{k/2} \rfloor$ and give an upper bound on the expected value.

Solution: $\binom{n}{k}2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \cdot 2^{1-\frac{k^2-k}{2}} = \frac{2^{\frac{k^2}{2}}}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} = \frac{2^{1+\frac{k}{2}}}{k!} < 1$

19: What happens if the upper bound is < 1?

Solution: There must be one an entry with value 0.

20: *Open problem* Find a nice proof that $r(4,5) = 25$.

21: *Open problem* Determine $r(5,5)$.

22: *Open problem* Determine (asymptotically) $r(k,k)$.